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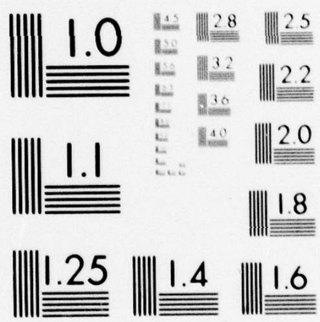

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A Maximum Likelihood Estimator for an Exponential Parameter from a Life Test with both Type I and Type II Censoring

by

Kenneth B. Fairbanks

Technical Report No. 86
Department of Statistics

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A hybrid life test on items assumed to have an exponential lifetime combines type I and type II censoring. In type I censoring, n items are placed on test and observed for a fixed time t*, while in type II censoring the test terminates with the r_o^{th} failure, where r_o is a preassigned integer. If t_{r_o} is the time of the r_o^{th} failure, a

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hybrid life test terminates at $\min(t_{r_0}, t^*)$. In some situations it may be of interest to estimate the average lifetime θ, following the test decision. In this report, we find the maximum likelihood estimator, $\hat{\theta}$, when the sample is subject to hybrid censoring. An expression for $E(\hat{\theta})$ is derived. Because this expression is complex, computer simulations are used to examine the bias of $\hat{\theta}$.

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Kenneth B. Fairbanks*
University of Missouri - Columbia

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This report is based on part of the author's Ph.D. dissertation.

University of Missouri
Department of Statistics
Columbia, Missouri 65211

A MAXIMUM LIKELIHOOD ESTIMATOR FOR AN EXPONENTIAL PARAMETER FROM A LIFE TEST WITH BOTH TYPE I AND TYPE II CENSORING

Abstract

A hybrid life test on items assumed to have an exponential lifetime combines type I and type II censoring. In type I censoring, n items are placed on test and observed for a fixed time t^*, while in type II censoring the test terminates with the r_o^{th} failure, where r_o is a preassigned integer. If t_{r_o} is the time of the r_o^{th} failure, a hybrid life test terminates at $\min(t_{r_o}, t^*)$. In some situations it may be of interest to estimate the average lifetime θ, following the test decision. In this report, we find the maximum likelihood estimator, $\hat{\theta}$, when the sample is subject to hybrid censoring. An expression for $E(\hat{\theta})$ is derived. Because this expression is complex, computer simulations are used to examine the bias of $\hat{\theta}$.

A MAXIMUM LIKELIHOOD ESTIMATOR FOR AN EXPONENTIAL PARAMETER FROM A LIFE TEST WITH BOTH TYPE I AND TYPE II CENSORING

1. Introduction

Collecting data for a life test is often complicated by some type of censoring on the observed lifetimes. Consequently, testing schemes involving various combinations of time and failure censoring have been developed. In type I censoring, n items are placed on test and observed for failures for a fixed time period t^*. In type II censoring, n items are placed on test and observed until the r_o^{th} failure occurs, where $r_o \leq n$. The value of r_o is chosen prior to the test. Epstein (1954) proposed a testing scheme which combines type I and type II censoring. In this scheme the test terminates at $\min(t_{r_o}, t^*)$, where t_{r_o} is the time of the r_o^{th} failure. We shall refer to this testing scheme as the hybrid test.

In this report we shall assume that the lifetimes under consideration have an exponential probability distribution. Thus, if τ represents the lifetime of an item, the density function of τ is

$$f(\tau;\theta) = \begin{cases} \frac{1}{\theta} e^{-\tau/\theta} & \text{if } \tau \geq 0 \\ 0 & \text{elsewhere} \end{cases} \tag{1.1}$$

where $\theta > 0$. In this form θ is the average lifetime of an item in the population.

Epstein (1954) developed the hybrid scheme to test $H_o: \theta = \theta_o$ against $H_1: \theta = \theta_1$, where $\theta_1 < \theta_o$. If $\min(t_{r_o}, t^*) = t^*$, H_o is accepted, while if $\min(t_{r_o}, t^*) = t_{r_o}$, H_o is rejected. Figure 1.1. illustrates this hybrid testing scheme.

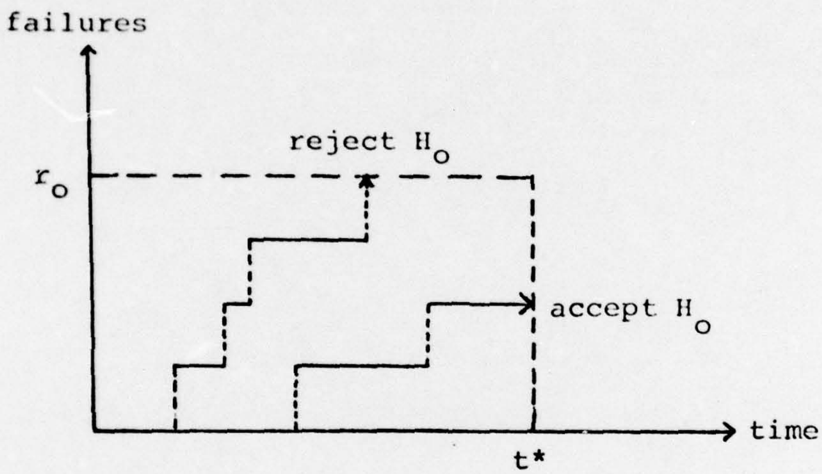

Figure 1.1. Possible sample paths in a hybrid scheme.

It is very likely that a true value of θ less than θ_1 may lead to rejection of H_o, or a true value of θ greater than θ_o may lead to the acceptance of H_o. Consequently, it may be of interest to compute an estimate of θ following the life test decision using the test data. The estimate might be a point estimate or an interval estimate of θ.

In this report we shall consider the problem of finding a point estimator, the maximum likelihood estimator of θ, when the data is collected for the hybrid life test. We shall assume that the life test is conducted without replacement, i.e. failed items are not replaced with new items.

Point estimation for θ under type I and type II censoring is thoroughly discussed in the literature of reliability and life testing. The maximum likelihood estimator of θ under type II censoring was found by Epstein and Sobel (1953). If $t_1, t_2, \ldots, t_n$ represent the ordered failure times of the n items on test, the maximum likelihood estimator of θ is given by

$$\hat{\theta} = \frac{1}{r_o}\left[\sum_{i=1}^{r_o} t_i + (n - r_o)t_{r_o}\right] = T_{r_o}/r_o \tag{1.2}$$

where $T_{r_o} = \sum_{i=1}^{r_o} t_i + (n - r_o)t_{r_o}$ represents the total accumulated test time, or total time on test, at the time of the r_o^{th} failure, t_{r_o}. The maximum likelihood estimator of θ under type I censoring, and its properties, are discussed by Bartholomew (1957), Mendenhall and Lehman (1960) and Bartholomew (1963), including a more general situation where each item on test has its own truncation time t_i^*, $i=1,2,\ldots,n$. Under type I censoring the maximum likelihood estimator is given by

$$\hat{\theta} = \frac{1}{k}\left[\sum_{i=1}^{k} t_i + (n - k)t^*\right] \tag{1.3}$$

where k is the number of failures observed by time t^*. These results and properties of the estimators are summarized in Mann, Schafer, and Singpurwalla (1974).

We may conjecture that (1.2) and (1.3) could be combined to form a maximum likelihood estimator under the hybrid censoring. This conjecture will be shown to be true in this report. For each item on test under this hybrid scheme, a mixed distribution is defined having positive probability at the point t^*. Order statistics from this distribution, which admit the possibility of ties, are defined and the likelihood of the first r_o order statistics is found. An expression for $E(\hat{\theta})$ is also derived in section II.2. A simple, closed expression for the bias is found when $r_o = 1$. A closed but complex expression for $E(\hat{\theta})$ is given for $2 \le r_o \le 5$. The bias of $\hat{\theta}$ is further examined by simulation, using some of the actual hybrid test schemes of Epstein (1954).

II. A MAXIMUM LIKELIHOOD ESTIMATOR FOR THE HYBRID TEST

II.1. Derivation of the maximum Likelihood Estimator

Our approach is to define a density for lifetimes under a hybrid test which admits ties with positive probability. The joint density of the first r_0 order statistics is then found, from which the maximum likelihood estimator is derived.

Let

$$f(t) = \begin{cases} \frac{1}{\theta} e^{-\frac{t}{\theta}} & 0 < t < t^* \\ e^{-\frac{t^*}{\theta}} & t = t^* \\ 0 & \text{elsewhere} \end{cases} \qquad (2.1)$$

Now define a measure μ as follows: If L is Lebesque measure and η is the counting measure, then for any Lebesque measurable set A,

$$\mu(A) = L(A) + \eta(A \cap \{t^*\}).$$

Then, if we define

$$P(A) = \int_A f(t)\,d\mu ,$$

$f(t)$ is a density in the sense that it is the Radon-Nikodyn derivative of P with respect to μ. If $\tau_1, \tau_2, \ldots, \tau_n$ are the n unordered failure times of the items in the hybrid test, we may consider $\tau_1, \ldots, \tau_n$ to be a random sample from a distribution with density function (2.1). Then the likelihood of θ is given by

$$L(\theta) = \prod_{i=1}^{n} f_i(\tau_i | \theta)$$

Let $0 < t_1 \leq t_2 \cdots \leq t_n \leq t^*$ define the corresponding order statistics. We have the possibility of ties here since an item may assume the value t^* (i.e., censored lifetimes) with probability $e^{-t^*/\theta}$. Consequently, the problem of these possible ties must be considered when finding the joint density of the order statistics. In the case of ties, the order statistics are defined uniquely, though somewhat artificially, in the following manner. If τ_i and τ_j are tied at t^* for the kth ordered failure time and $i < j$, define $t_k = \tau_i$ and $t_{k+1} = \tau_j$. In most considerations of order statistics, the density is assumed to be continuous, and points in n-space where two or more ties occur are ignored since they have zero probability. Now these points must be considered separately. To that end, we partition n-space into sets $A_0, A_1, \ldots, A_n$, where we define $A_i = \{\tau_1, \ldots, \tau_n$: exactly i points equal $t^*\}$.

Thus, A_0 is the set where all coordinates differ from t*, while A_3, for example, is the set of points where three coordinates equal t*. The joint density of the order statistics will differ on each A_i. We further partition each A_i to obtain the required one-to-one transformation onto the set

$$\{(t_1, t_2, \ldots, t_n); 0 < t_1 \leq t_2 \leq \cdots \leq t_n \leq t^*\} .$$

For example, when n = 3, A_2 is partitioned into

$$a_{21} = \{(\tau_1, \tau_2, \tau_3); 0 < \tau_1 < \tau_2 = \tau_3 = t^*\}$$
$$\Rightarrow t_1 = \tau_1, t_2 = \tau_2, t_3 = \tau_3,$$

$$a_{22} = \{(\tau_1, \tau_2, \tau_3); 0 < \tau_2 < \tau_1 = \tau_3 = t^*\}$$
$$\Rightarrow t_1 = \tau_2, t_2 = \tau_1, t_3 = \tau_3,$$

and

$$a_{23} = \{(\tau_1, \tau_2, \tau_3); 0 < \tau_3 < \tau_1 = \tau_2 = t^*\}$$
$$\Rightarrow t_1 = \tau_3, t_2 = \tau_1, t_3 = \tau_2 .$$

Consequently, on A_2 with n = 3, the joint density of the order statistics is

$$g(t_1, t_2, t_3) = 3f(t_1)f(t_2)f(t_3) .$$

In general, there will be $\binom{n}{i}(n-i)!$ sets in the partition of A_i to accomplish the one-to-one transformation. $\binom{n}{i}$ accounts for the number of ways to select the i ties, while the (n-i) untied coordinates can be permuted (n-i)! ways. The joint density for the order statistics over set A_i can be written as

$$g(t_1, \ldots, t_n) = \binom{n}{i}(n-i)! \prod_{j=1}^{n} f(t_j), \; i = 0, 1, \ldots, n.$$

If we define $C_{n,i} = \binom{n}{i}(n-i)! = n!/i!$ and let $q = e^{-t^*/\theta}$, then the joint density of the order statistics, in full generality, is written as

$$g(t_1,\ldots,t_n) = \begin{cases} C_{n,0} \prod_{i=1}^{n} f(t_i) & \text{if } 0<t_1<t_2<\cdots<t_n < t^* \\ C_{n,1}\, q \prod_{i=1}^{n-1} f(t_i) & \text{if } 0<t_1<t_2<\cdots<t_n = t^* \\ C_{n,2}\, q^2 \prod_{i=1}^{n-2} f(t_i) & \text{if } 0<t_1<t_2<\cdots<t_{n-1} = t_n = t^* \\ \vdots & \\ C_{n,n} q^n & \text{if } 0<t_1 = t_2 = \cdots = t_n = t^*. \end{cases}$$

It is now possible to find the joint marginal density of the first r_0 out of n order statistics from density (2.1). Recall that r_0 is the truncation value on the number of failures for the hybrid test. This joint marginal density can be found from the theorem which follows.

Theorem 2.1. If f(t) represents the density function (2.1) and

$$F(t) = \int_0^t f(t)\,d\mu ,$$

then the joint marginal density function of the first (n-r) order statistics having density f(t) is

$$g(t_1,\ldots,t_{n-r}) = \begin{cases} C_{n,r}[1-F(t_{n-r})]^r \prod_{i=1}^{n-r} f(t_i) \\ \text{if } 0 < t_1 < \cdots < t_{n-r} < t^* \\[2ex] C_{n,r+1}\, q^{r+1} \prod_{i=1}^{n-r-1} f(t_i) \\ \text{if } 0 < t_1 < \cdots < t_{n-r} = t^* \\[2ex] C_{n,r+2}\, q^{r+2} \prod_{i=1}^{n-r-2} f(t_i) \\ \text{if } 0 < t_1 < \cdots < t_{n-r-1} = t_{n-r} = t^* \\ \vdots \end{cases}$$

$$\left| C_{n,n}\, q^n \text{ if } 0<t_1 = t_2 = \cdots\ t_{n-r} = t^*. \right.$$

The proof is by induction on r, for $r \leq n-1$. Consider the case where $r = 1$. In the region where $0 < t_1 < \cdots < t_n \leq t^*$,

$$g(t_1, \ldots, t_{n-1}) = C_{n,0} \int_{t_n=t_{n-1}}^{t^*-} \prod_{i=1}^{n} f(t_i)\,dt_n +$$

$$C_{n,1} \int_{\{t^*\}} \prod_{i=1}^{n-1} f(t_i)\,q\,d\mu \ ,$$

where the limits on the first integral indicate that integration is over the interval $[t_{n-1}, t^*)$, while the second integral is over the single point t^*.

Then,

$$g(t_1, \ldots, t_{n-1}) = C_{n,0} \left[F(t_n) \right]_{t_{n-1}}^{t^*-} \prod_{i=1}^{n-1} f(t_i) + C_{n,1}\, q \prod_{i=1}^{n-1} f(t_i)$$

$$= C_{n,0}[1-q-F(t_{n-1})] \prod_{i=1}^{n-1} f(t_i) + C_{n,1}\, q \prod_{i=1}^{n-1} f(t_i)$$

$$= C_{n,1}[1-F(t_{n-1})] \prod_{i=1}^{n-1} f(t_i) \text{ if } 0<t_1< \cdots <t_{n-1}<t^*.$$

The last equality follows from the observation that $C_{n,0} = C_{n,1}$. Now in the region where

$$0 < t_1 < \cdots < t_{n-1} = t_n = t^* ,$$

the integration will be over the single point t^*. Thus,

$$g(t_1, \ldots, t_{n-1}) = C_{n,2} \int_{\{t^*\}} \prod_{i=1}^{n-2} f(t_i) q^2 d\mu$$

$$= C_{n,2}\, q^2 \prod_{i=1}^{n-2} f(t_i)$$

$$\text{if } 0 < t_1 < t_2 < \cdots < t_{n-1} = t^*.$$

Likewise,

$$g(t_1,\ldots,t_{n-1}) = C_{n,3} q^3 \prod_{i=1}^{n-3} f(t_i) \quad \text{if } 0<t_1<t_2<\cdots<t_{n-2}=t_{n-1}=t^*$$

$$\vdots$$

$$g(t_1,\ldots,t_{n-1}) = C_{n,n} q^n \qquad \text{if } 0<t_1=t_2=\cdots=t_{n-1}=t^*,$$

which is the desired result for $r = 1$. Now assume the theorem is true for $r = k$. It suffices to show it also holds for $r = k + 1$. Again in the region

$$0 < t_1 < \cdots < t_{n-k} \leq t^*,$$

$$g(t_1, \ldots, t_{n-k-1})$$

$$= C_{n,k} \left(\prod_{i=1}^{n-k-1} f(t_i) \right) \int_{t_{n-k}=t_{n-k-1}}^{t^*-} f(t_{n-k}) [1-F(t_{n-k})]^k dt_{n-k}$$

$$+ C_{n,k+1} \int_{\{t^*\}} \prod_{i=1}^{n-k-1} f(t_i) q^{k+1} d\mu .$$

Carrying out the integration we find

$$g(t_1, \ldots, t_{n-k-1})$$

$$= C_{n,k} \left[-\frac{(1-F(t_{n-k}))^{k+1}}{k+1} \right]_{t_{n-k-1}}^{t^*} \prod_{i=1}^{n-k-1} f(t_i)$$

$$+ C_{n,k+1} q^{k+1} \prod_{i=1}^{n-k-1} f(t_i)$$

$$= \frac{C_{n,k}}{k+1} \{[1-F(t_{n-k-1})]^{k+1} - q^{k+1}\} \prod_{i=1}^{m-k-1} f(ti)$$

$$+ C_{n,k+1} q^{k+1} \prod_{i=1}^{n-k-1} f(t_i)$$

$$= C_{n,k+1}[1-F(t_{n-k-1})]^{k+1} \prod_{i=1}^{n-k-1} f(t_i),$$

$$\text{if } 0 < t_1 < \cdots < t_{n-k-1} < t^*.$$

Again, the last equality follows from the fact

$$C_{n,k}/k+1 = C_{n,k+1} .$$

Then in the region where $0 < t_1 < \cdots < t_{n-k-1} = t_{n-k} = t^*$,

$$g(t_1, \ldots, t_{n-k-1}) = \int_{\{t^*\}} C_{n,k+2} \prod_{i=1}^{n-k-2} f(t_i) q^{k+2} d\mu$$

$$= C_{n,k+2}\, q^{k+2} \prod_{i=1}^{n-k-2} f(t_i)$$

$$\text{if } 0 < t_1 < \cdots < t_{n-k-1} = t^*.$$

Likewise,

$$g(t_1, \ldots, t_{n-k-1}) = C_{n,k+3}\, q^{k+3} \prod_{i=1}^{n-k-3} f(t_i)$$

$$\text{if } 0 < t_1 < \cdots < t_{n-k-2} = t_{n-k-1} = t^*$$

$$\vdots$$

$$g(t_1, \ldots, t_{n-k-1}) = C_{n,n} q^n$$

$$\text{if } 0 < t_1 = t_2 = \cdots = t_{n-k-1} = t^* .$$

This establishes the result for $r = k + 1$ and the proof is complete by induction.

From Theorem 2.1 , the joint marginal density of the first r_0 order statistics is seen to be

$$g(t_1, \ldots, t_{r_0}) = \begin{cases} C_{n,n-r_0}[1-F(t_{r_0})]^{n-r_0} \prod_{i=1}^{r_0} f(t_i) \\ \text{if } 0 < t_1 < \cdots < t_{r_0} < t^* \\ \\ C_{n,n-r_0+1} q^{n-r_0+1} \prod_{i=1}^{r_0-1} f(t_i) \\ \text{if } 0 < t_1 < \cdots < t_{r_0} = t^* \\ \\ C_{n,n-r_0+2} q^{n-r_0+2} \prod_{i=1}^{r_0-2} f(t_i) \\ \text{if } 0 < t_1 < \cdots < t_{r_0-1} = t_{r_0} = t^* \\ \vdots \\ C_{n,n} q^n \\ \text{if } 0 < t_1 = t_2 = \cdots = t_{r_0} = t^* \end{cases}$$

Equivalently, we can write

$$g(t_1, \ldots, t_{r_0}) = \begin{cases} C_{n,n-r_0}[1-F(t_{r_0})]^{n-r_0} \prod_{i=1}^{r_0} f(t_i) & \text{if } t_{r_0} < t^* \\ \\ C_{n,n-k} q^{n-k} \prod_{i=1}^{k} f(t_i) & \text{if } t_{r_0} \geq t^*, \end{cases}$$

where k = the number of failures observed in $(0,t^*)$. Note that $0 \leq k \leq r_0$ and $k = r_0$ if and only if $t_{r_0} < t^*$.

Consequently we may write the likelihood function of the first r_0 order statistics as

$$L(\theta) \propto [\prod_{i=1}^{k} f(t_i)] q^{(1-a)(n-k)} [1-F(t_{r_0})]^{a(n-r_0)} \qquad (2.2)$$

where

$$a = \begin{cases} 1 \text{ if } t_{r_0} < t^* \\ 0 \text{ if } t_{r_0} \geq t^* . \end{cases}$$

In terms of density function (2.2), the likelihood becomes

$$L(\theta) \propto \frac{1}{\theta^k} e^{-\frac{1}{\theta} \sum_{i=1}^{k} t_i} e^{-\frac{t^*}{\theta}(1-a)(n-k)} e^{-\frac{t_{r_0}}{\theta}(n-r_0)a}$$

$$= \frac{1}{\theta^k} e^{-\frac{1}{\theta}[\sum_{i=1}^{k} t_i + (1-a)(n-k)t^* + a(n-r_0)t_{r_0}]} .$$

Then,

$$\ln L(\theta) = C - k\ln\theta - \frac{1}{\theta}[\sum_{i=1}^{k} t_i + (1-a)(n-k)t^* + a(n-r_0)t_{r_0}]$$

and

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{-k}{\theta} + \frac{[\sum_{i=1}^{k} t_i + (1-a)(n-k)t^* + a(n-r_0)t_{r_0}]}{\theta^2}$$

Setting

$$\frac{\partial \ln L(\theta)}{\partial \theta} = 0,$$

we find that the maximum likelihood estimator for θ is

$$\hat{\theta} = \frac{\sum_{i=1}^{k} t_i + (1-a)(n-k)t^* + a(n-r_0)t_{r_0}}{k} \quad . \qquad (2.3)$$

This is merely the combination of the maximum likelihood estimators from the type I and type II censoring which we expected.

II.2 The First Moment of the Maximum Likelihood Estimator

The maximum likelihood estimator derived in the previous section can be used to estimate θ following the hybrid life test. In order to examine the bias of this estimator, an expression is derived in this section for the first moment of $\hat{\theta}$. In section II.1 we obtained

$$\hat{\theta} = \frac{1}{k} \{ \sum_{i=1}^{k} t_i + (1-a)(n-k)t^* + a(n-r_0)t_{r_0} \},$$

where t_i = i^{th} ordered failure time, $i = 1, 2, \ldots, n$,

t^* = truncation value for time,

r_0 = truncation value for failures, and

k = number of failures at time of decision.

The distribution of k is given by

$$P(k = j) = \begin{cases} \binom{n}{j} p^j (1-p)^{n-j} & \text{if } j = 0, 1, \ldots, r_0 - 1 \\ \sum_{i=r_0}^{n} \binom{n}{i} p^i (1-p)^{n-i} & \text{if } j = r_0, \end{cases}$$

where $p = P(\text{an item fails before } t^*) = 1 - e^{-t^*/\theta}$. In (2.3), "a" is a random indicator with a distribution given by $P(a=1) = P(k=r_0) = P(t_{r_0} < t^*)$, and $P(a=0) =$

$1-P(a=1)$.

The estimator $\hat{\theta}$ will be infinte when $k = 0$, so we will condition on $\hat{\theta}$ being finite and find $E(\hat{\theta}|\hat{\theta}<\infty) = E(\hat{\theta}|k>0)$. Since $\hat{\theta}$ is infinite with probability $P(k=0)$, the distribution of $\hat{\theta}$ conditional on $\hat{\theta} < \infty$ is just the unconditional distribution of $\hat{\theta}$ divided by the constant $P(k>0)$. Hence, $E(\hat{\theta}|\hat{\theta}<\infty)$ will be merely the unconditional expectation of $\hat{\theta}$ (over finite values of $\hat{\theta}$) divided by the constant $P(k>0)$. So, $E(\hat{\theta}|\hat{\theta}<\infty) =$

$$\frac{1}{P(k>0)}\left\{E\left[\frac{\sum_{i=1}^{k} t_i}{k}\right] + E\left[\frac{(1-a)(n-k)t^*}{k}\right] + E\left[\frac{a(n-r_0)t_{r_0}}{k}\right]\right\} \quad (2.4)$$

Each of the expectations in (2.4) will be evaluated in turn. We first write

$$E\left[\frac{\sum_{i=1}^{k} t_i}{k}\right] = \sum_{j=1}^{r_0} E\left[\frac{\sum_{i=1}^{j} t_i}{j}\,\middle|\, k=j\right] P(k=j).$$

Given that $k = j$, for $j = 1, 2, \ldots, r_0-1$, there are exactly $\binom{n}{j}$ equally likely ways of selecting the set of j items out of n which will fail first. If we condition again on each of the $\binom{n}{j}$ equally likely combinations it is seen that

$$E\left[\frac{\sum_{i=1}^{k} t_i}{k}\,\middle|\, k=j\right] = E\left[\frac{\sum_{i=1}^{k} \tau_i}{k}\,\middle|\, k=j\right] = E\left[\frac{\sum_{i=1}^{j} \tau_i}{j}\right],$$

where τ_i is the i^{th} unordered failure time whose distribution is truncated at t^*. The truncated distribution follows from the fact that we are given the number of failures observed before time t^* is $k = j$. Thus the distribution of τ_i is given by

$$f(\tau_i) = \begin{cases} \dfrac{\frac{1}{\theta} e^{-\tau_i/\theta}}{1-e^{-t^*/\theta}} = \dfrac{\frac{1}{\theta} e^{-\tau_i/\theta}}{p} & \text{if } 0 < \tau_i < t^* \\ 0 & \text{elsewhere.} \end{cases} \qquad (2.5)$$

Direct integration using (2.5) gives the result that $E(\tau_i) = \frac{\theta - qt^*}{p}$, where $q = 1-p$. So,

$$E\left[\sum_{i=1}^{j} \frac{t_i}{j} \middle| k=j\right] = \frac{\theta - qt^*}{p}, \text{ for } j = 1, 2, \ldots, r_0-1.$$

Now, $k = r_0$ implies that at least r_0 items failed by time t^* (even though observation ceases at t_{r_0}). We must treat this situation differently.

$$E\left[\sum_{i=1}^{k} \frac{t_i}{k} \middle| k=r_0\right] = \frac{1}{r_0} E\left[\sum_{i=1}^{r_0} t_i \middle| t_{r_0} < t^*\right].$$

To evalute this expectation we need the joint density of the first r_0 order statistics taken from the exponential population, conditioned on $t_{r_0} < t^*$. Let x_i represent the observed value of the random time t_i. Then

$$F_{t_1, \ldots, t_{r_0}}(x_1, \ldots, x_{r_0} | t_{r_0} < t^*)$$

$$= \frac{p[t_1 \le x_1, t_2 \le x_2, \ldots, t_{r_0} \le \min(x_{r_0}, t^*)]}{P(t_{r_0} \le t^*)},$$

and

$$f_{t_1, \ldots, t_{r_0}}(x_1, \ldots, x_{r_0} | t_{r_0} < t^*)$$

$$= \frac{\partial F_{t_1, \ldots, t_{r_0}}(x_1, \ldots, x_{r_0} | t_{r_0} < t^*)}{\partial x_1 \partial x_2 \ldots \partial x_{r_0}}$$

$$= \begin{cases} \dfrac{f_{t_1, \ldots, t_{r_0}}(x_1, x_2, \ldots, x_{r_0})}{P(t_{r_0} < t^*)} & \text{if } t_{r_0} < t^* \\ 0 & \text{elsewhere.} \end{cases}$$

Since

$$f_{t_1, \ldots, t_{r_0}}(x_1, \ldots, x_{r_0}) = \frac{n!}{(n-r_0)!\,\theta^{r_0}} e^{-\frac{1}{\theta}\left[\sum_{i=1}^{r_0} x_i + (n-r_0) x_{r_0}\right]},$$

we can write

$$E\left[\sum_{i=1}^{k} \frac{t_i}{k} \middle| k = r_0\right]$$

as

$$\frac{1}{r_0 P(t_{r_0} < t^*)} \int_{x_1=0}^{t^*} \int_{x_2=x_1}^{t^*} \cdots \int_{x_{r_0}=x_{r_0-1}}^{t^*} \left(\sum_{i=1}^{r_0} x_i\right) \frac{n!}{(n-r_0)!\theta^{r_0}}$$

$$e^{-\frac{1}{\theta}\left[\sum_{i=1}^{r_0} x_i + (n-r_0)x_{r_0}\right]} dx_{r_0} \cdots d_{x_1} \qquad (\ 2.6\ )$$

Evaluating and simplifying this r_0-fold integral becomes very tedious as r_0 increases. We have evaluated (2.6) for values of r_0 through five. We give here the results for $r_0 = 1$ and 2, while the results for $r_0 = 3$, 4, and 5 are given in Appendix 1. In these expressions we define

$$C_r = \frac{n!}{(n-r)!\,rP(t_r<t^*)} \ . \qquad (\ 2.7\ )$$

The integral (2.6) reduces to

$$C_1\left[\frac{\theta}{n^2} - \left(\frac{\theta+nt^*}{n^2}\right) e^{-\frac{nt^*}{\theta}}\right]$$

when $r_0 = 1$, and

$$C_2\left[\frac{\theta(3n-2)}{n^2(n-1)^2} + \left(\frac{2nt^*+(n+2)\theta}{n^2}\right)e^{-\frac{nt^*}{\theta}} - \left(\frac{n\theta+(n-1)t^*}{(n-1)^2}\right)e^{-\frac{(n-1)t^*}{\theta}}\right]$$

when $r_0 = 2$.

We now have, for the first expectation in expression (2.4)

$$E\left[\sum_{i=1}^{k} \frac{t_i}{k}\right] = \sum_{i=1}^{r_0-1} \frac{(\theta-qt^*)}{p} P(k=i) + E\left[\sum_{i=1}^{k} \frac{t_i}{k} \Big| k=r_0\right] P(k=r_0)$$

$$= \frac{P(0<k<r_0)(\theta-qt^*)}{p} + E\left[\sum_{i=1}^{k} \frac{t_i}{k} \Big| k=r_0\right] P(k=r_0) \qquad (2.8)$$

where

$$E\left[\sum_{i=1}^{k} \frac{t_i}{k} \Big| k=r_0\right]$$

is given by (2.6).

Next, we evaluate

$$E\left[\frac{(1-a)(n-k)t^*}{k}\right] .$$

If we condition on the value of a, this expectation becomes

$$E\left[\frac{(n-k)t^*}{k} \Big| a=0\right] P(a=0) =$$

$$E\left[\frac{nt^*}{k} \Big| k<r_0\right] P(k<r_0) - t^*P(k<r_0) =$$

$$nt^* \sum_{i=1}^{r_0-1} \left(\frac{1}{i}\right) P(k=i) - t^*P(k<r_0) =$$

$$t^* \sum_{i=1}^{r_0-1} \frac{(n-i)}{i} P(k=i) \ .$$

To establish this, we made use of the fact that a = 0 if and only if $k < r_0$. As before,

$$P(k=i) = \binom{n}{i}\left(1-e^{-\frac{t^*}{\theta}}\right)^i \left(e^{-\frac{t^*}{\theta}}\right)^{n-i} \ .$$

Finally, we find

$$(n-r_0)\ E\left[\frac{at_{r_0}}{k}\right].$$

Conditioning again on the value of a, we get

$$E\left[\frac{at_{r_0}}{k}\right] = E\left[\frac{t_{r_0}}{k}\middle| a=1\right]P(a=1) = E\left[\frac{t_{r_0}}{r_0}\middle| t_{r_0}<t^*\right]P(t_{r_0}<t^*) \ ,$$

since $a = 1 \leftrightarrow t_{r_0} < t^* \leftrightarrow k = r_0$. Here we require the density of the r_0^{th} order statistic from the exponential distribution which is truncated at t^*.

$$f_{t_{r_0}}(x_{r_0} | t_{r_0} < t^*) = \begin{cases} \dfrac{n!\left(1-e^{-\frac{x_{r_0}}{\theta}}\right)^{r_0-1} e^{-\frac{(n-r_0+1)x_{r_0}}{\theta}}}{P(t_{r_0} < t^*)\theta(n-r_0)!(r_0-1)!} \\ \text{if } x_{r_0} < t^* \\ 0 \quad \text{elsewhere.} \end{cases}$$

Then

$$E\left[\frac{t_{r_0}}{r_0} \middle| t_{r_0} < t^*\right] P(t_{r_0} < t^*) =$$

$$\frac{1}{r_0} \int_{t=0}^{t^*} \frac{n!\,t\theta^{-1}}{(n-r_0)!\,(r_0-1)!} (1-e^{-\frac{t}{\theta}})^{r_0-1} e^{\frac{-(n-r_0+1)t}{\theta}} dt$$

$$= \frac{n\binom{n-1}{r_0-1}}{r_0\theta} \int_0^{t^*} t(1-e^{-\frac{t}{\theta}})^{r_0-1} e^{\frac{-(n-r_0+1)t}{\theta}} dt. \quad (2.9)$$

But

$$\left(1-e^{-\frac{t}{\theta}}\right)^{r_0-1} = \sum_{j=0}^{r_0-1} \binom{r_0-1}{j} e^{-\frac{jt}{\theta}} (-1)^j ,$$

by the binomial expansion. The integral (2.9) becomes

$$\frac{n\theta\binom{n-1}{r_0-1}}{r_0} \sum_{j=0}^{r_0-1} (-1)^j \binom{r_0-1}{j} \int_0^{t^*} \frac{t}{\theta^2} e^{-(n-r_0+j+1)t^*/\theta} dt$$

$$= \frac{n\theta\binom{n-1}{r_0-1}}{r_0} \sum_{j=0}^{r_0-1} \frac{(-1)^j\binom{r_0-1}{j}}{(n-r_0+j+1)^2} \left[1-e^{\frac{-(n-r_0+j+1)t^*}{\theta}}\left(\frac{(n-r_0+j+1+t^*}{\theta}+1\right)\right].$$

Combining the three parts from (2.4) we have a final expression for the first moment of $\hat{\theta}$, conditional on $\hat{\theta}$ being finite.

$$E(\hat{\theta}|\hat{\theta}<\infty) = \frac{P(0<k<r_0)(\theta-qt^*)}{P(k>0)p} + \frac{P(k=r_0)}{r_0 P(k>0)} E\left[\sum_{i=1}^{r_0} t_i \middle| t_{r_0} < t^*\right]$$

$$+ \frac{t^*}{P(k>0)} \sum_{i=1}^{r_0-1} P(k=i) \frac{(n-i)}{i} +$$

$$\frac{(n-r_0)n\theta\binom{n-1}{r_0-1}}{r_0 \; P(k>0)} \sum_{j=0}^{r_0-1} \frac{(-1)^j\binom{r_0-1}{j}}{(n-r_0+j+1)^2} \times$$

$$\left[1-e^{-\frac{(n-r_0+j+1)t^*}{\theta}}\left(\frac{(n-r_0+j+1)t^*}{\theta} + 1\right)\right]. \qquad (2.10)$$

In the simplest state, when $r_0 = 1$, $E(\hat{\theta}|k>0)$ reduces to

$$\frac{1}{P(k>0)} [\theta-(\theta+nt^*)e^{-nt^*/\theta}],$$

where $P(k>0) = P(t_1 < t^*) = 1-e^{-nt^*/\theta} = p^*$, say.

So,

$$E(\hat{\theta}|k>0) = \frac{\theta-\theta(1-p^*)-nt^*(1-p^*)}{p^*} = \theta-nt^*(\frac{1}{p^*} -1).$$

When $r_0 = 1$, $\hat{\theta}$ has a negative bias equal to $nt^*(\frac{1}{p^*} - 1)$. Similar simplifications when $r_0 > 1$ were not found.

If this maximum likelihood estimate of θ is computed with the data from a hybrid life test, the values of n, r_0, and t^* are supplied by the test and cannot be adjusted for the purpose of reducing the bias. We have simulated some hybrid life tests on the computer for the purpose of observing some bias values which the maximum likelihood estimator inherits from the test. If we test $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, where $\theta_1 < \theta_0$, Epstein (1954) has given the theory necessary to find the values of n, r_0 and t^* which correspond to given error probabilities α and β. Simulations, of 1000 trials each, were run on 27 different hybrid test schemes. In each simulation $E(\hat{\theta})$ was estimated and the bias was estimated at $\theta = \theta_0$. Table 2.1 gives the simulation results. The bias is expressed in the table as the percentage of the actual value of θ. Using 1000 trials, the estimated standard deviation for this percentage was less than 3% for all simulations. For comparison, the table also gives the exact bias in each case for the maximum likelihood estimate from a type I testing scheme having the same values of n and t^*. This exact bias was shown by Bartholomew (1957) to be

TABLE 2.1

Bias of the M.L.E. from Some Hybrid Tests

θ_0/θ_1	α = .01			α = .05			α = .10		
	β=.01	β=.05	β=.10	β=.01	β=.05	β=.10	β=.01	β=.05	β=.10
2	46	35	30	33	23	19	26	18	15
	120	87	72	90	59	48	73	48	39
	2.48	2.96	4.17	3.38	4.58	6.22	3.06	6.17	8.18
	2.60	3.66	4.50	3.53	5.63	7.13	4.44	7.13	9.16
3	19	15	13	13	10	8	11	8	6
	41	30	26	30	21	16	27	18	13
	8.36	13.06	15.79	13.72	18.31	23.74	13.62	21.55	27.76
	8.67	12.92	15.62	12.92	20.50	26.90	14.86	24.27	29.98
5	9	8	7	7	5	4	5	4	3
	15	13	11	13	8	6	10	8	5
	22.80	27.49	30.73	28.25	23.04	8.18	27.40	20.82	-2.41
	28.08	29.96	30.14	29.96	23.23	9.74	29.08	23.23	-1.07

Note: (1). t^* is such that $\theta_0/t^* = 3$ for all cases and $\theta_1 = 1$.

(2). The first two entries in each cell are the values of r_0 and n needed to give the indicated values of α and β for that cell.

(3). The third entry is the simulation estimate of $\frac{E(\hat{\theta})-\theta_0}{\theta_0} \times 100$.

(4). The final entry is the exact bias from the corresponding type I scheme.

$$nt^*E(\frac{1}{k}) - \frac{qt^*}{p} - t^*$$

where k = the number of failures in $(0,t^*)$. It is assumed that $k > 0$ and $E(1/k)$ is conditional on $k > 0$. As before, $q = e^{-t^*/\theta}$ and $p = 1-q$. It is clear from Table 2.1 that the bias may be quite large especially for smaller sample sizes.

APPENDIX 1

Listed here are the values of the integral (2.6) when r_0 = 3, 4, or 5. The constant Cr is defined in equation (2.7).

$$r_0 = 3: \quad C_3 \left[\frac{2\theta(3n^2-7n+3)}{n^2(n-1)^2(n-2)^2} - \frac{2n\theta+3\theta+3nt^*}{2n^2} e^{\frac{-nt^*}{\theta}} + \frac{2n\theta+2(n-1)t^*}{(n-1)^2} e^{\frac{-(n-1)t^*}{}} - \frac{2n\theta-3\theta+(n-2)t^*}{2(n-2)^2} e^{\frac{-(n-2)t^*}{\theta}} \right]$$

$$r_0 = 4: \quad C_4 \left[\frac{2\theta(5n^3-25n^2+35n-12)}{n^2(n-1)^2(n-2)^2(n-3)^2} + \frac{3n\theta+4nt^*+\theta}{6n^2} e^{\frac{-nt^*}{\theta}} - \frac{3n\theta+3(n-1)t^*}{2(n-1)^2} e^{\frac{-(n-1)t^*}{\theta}} + \frac{3n\theta-4\theta+2(n-2)t^*}{2(n-2)^2} e^{\frac{-(n-2)t^*}{\theta}} - \frac{3n\theta-8\theta+(n-3)t^*}{6(n-3)^2} e^{\frac{-(n-3)t^*}{\theta}} \right]$$

$$r_0 = 5: \quad C_5 \left[\theta\frac{(15n^4-130n^3+375n^2-404n+120)}{n^2(n-1)^2(n-2)^2(n-3)^2(n-4)^2} - \frac{4n\theta+5nt^*+5\theta}{24n^2} e^{\frac{-nt^*}{\theta}} + \frac{2n\theta+2(n-1)t^*}{3(n-1)^2} e^{\frac{-(n-1)t^*}{\theta}} \right.$$

$$- \frac{4n\theta-5\theta+3(n-2)t^*}{4(n-2)^2} e^{\frac{-(n-2)t^*}{\theta}}$$

$$+ \frac{2n\theta-5\theta+(n-3)t^*}{3(n-3)^2} e^{\frac{-(n-3)t^*}{\theta}}$$

$$- \left.\frac{4n\theta-15\theta+(n-4)t^*}{24(n-4)^2} e^{\frac{-(n-4)t^*}{\theta}}\right]$$

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